A SIMPLE MODEL TO DETERMINE THE ALTITUDE AND VELOCITY OF THE I.S.S.

HOW CAN SIMPLE MEASUREMENTS OF THE ORBITAL PERIOD OF A SATELLITE BE USED TO CALCULATE ITS ALTITUDE AND VELOCITY?

16 March 2016

The report is twelve pages long, from the *Introduction* to the *Evaluation & Conclusion.*

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ACKNOWLEDGEMENTS

To the *Observatoire de Lausanne,* for allowing us to use their observation grounds to take our measurements; to members of the CERN labs, for allowing us to borrow one of their old theodolites for our experiment; and to my friends, without whom I would never have been able to carry on through the long nights taking measurements, and who were willing to put up and comply with the irregular schedules of the I.S.S.'s visible passes.

Introduction

According to the Laws of gravitation, it is possible to determine the radius of orbit of a satellite from its period of orbit so long as one knows the mass of the parent body. From this, the research question of this report is *'How can simple measurements of the orbital period of a satellite be used to calculate its altitude and velocity?'* The satellite selected is the International Space Station (I.S.S.), chosen for being the "brightest man-made object in the sky excluding flares"[1] when it lights up due to the reflection of sunlight from its solar array. It is the largest human-built satellite, being 108,5 metres wide and $450 \cdot 10^9$ kilogrammes in mass.

My interest in orbital mechanics was piqued by the enthusiasm of the substitute teacher who taught us this topic. Orbital mechanics are an integral part of any space-related physics and can for example be used to calculate the distance to the moon. This experiment allows me develop an appreciation of the difficulties and methods of data collection and of scientists' use of modelling.

In this report, we will first consider the theories of orbital mechanics necessary for the calculations. Then, we will turn to the experimental method we used to collect our data. Subsequently, we will use a simple model, that of a perfectly regular circular orbit, to calculate the radius (which gives the altitude) and velocity of orbit of the I.S.S. from our data; the result for orbital velocity will then be re-evaluated using a second, more complex model. The results of both sets of calculations will finally be compared to their literature values and, in conclusion, the experiment will be reviewed and evaluated.

Symbols

- G [$\text{N m}^2\text{ kg}^{-2}$] – the universal gravitational constant (literature value).
- T [s] – the period of orbit of the I.S.S. (measured).
- $\vec{\omega}$ [rad s^{-1}] – the angular velocity of the I.S.S. (calculated from equation $\vec{\omega} = \frac{2\pi}{T}$).
- $\vec{v}$ [m s^{-1}] – the linear velocity of the I.S.S. (calculated).
- r [m] – the radius of orbit of the I.S.S. (calculated).
- $m_\oplus$ [kg] and $r_\oplus$ [m] – respectively the mass and radius of the earth (literature values).
- h [m] – the altitude or height of the I.S.S. above the earth (calculated from $h = r - r_\oplus$).
- θ_n – the angular position in the sky of the I.S.S. at a given point, composed of V and Az (measured).
- V [^{g}] – the vertical angle between the zenith[2] and the I.S.S. (measured by the theodolite in its native units of grades, where $400^g = 2\pi$ rad).
- Az [^{g}] – the azimuthal (horizontal) angle between due north and the I.S.S. (measured).

[1] (Wikimedia Foundation, Inc., 2015)

[2] "Point of heavens directly above observer (opp. NADIR)." (Oxford University Press, 1982 p. 1253)

Orbital mechanics

The calculations for the simplest possible model, that of a perfectly circular and regular orbit, are mainly derived from two standard theorems of classical mechanics: Newton's theory of gravitation and the law of required force for stable circular motion.[3] These are respectively:

$$\vec{F}_g = G\frac{m_\oplus m}{r^2} \text{ and } \vec{F}_c = m\vec{\omega}^2 r$$

Equating these and solving for r gives:

$$G\frac{m_\oplus}{r^2} = \vec{\omega}^2 r \;;\; G\frac{m_\oplus}{\vec{\omega}^2} = r^3$$

$$r = \sqrt[3]{G\frac{m_\oplus}{\vec{\omega}^2}}$$

When undergoing circular motion, the velocity $\vec{v}$ is equal to the angular velocity $\vec{\omega}$ in radians per second, multiplied by the orbital radius r in metres. Thus:

$$\vec{v} = \vec{\omega}r = \vec{\omega}\cdot\sqrt[3]{G\frac{m_\oplus}{\vec{\omega}^2}} = \sqrt[3]{G\vec{\omega}m_\oplus}$$

This gives expressions we can use to calculate $\vec{v}$ and $\vec{r}$ based on a measured variable T, the constant G, and the mass of the earth $m_\oplus$ which we assume to be constant for the purposes of this report.

Apparatus and experimental method

The first part of the investigation was to measure the period of orbit of the satellite. For this, we used a chronometer accurate to a tenth of a second. We activated it when the I.S.S. was visually at its highest point (with the smallest observed 'vertical angle') on two consecutive sightings in order to evaluate the time between them. Confusion with other satellites was avoided through the use of the online data base *Heavens-Above*,[4] by the intermediary of a mobile-phone application.[5]

However, we realised after taking a test reading that the I.S.S. does not execute a perfectly regular circular orbit around the earth: during two consecutive sightings, it appeared once to the north of us and then to the south. This discrepancy will be somewhat compensated for with a more complex model in the section More complex modelling. Although the calculations required for the more complex model will be discussed later, they involve knowing the angular position of the I.S.S. with respect to the observer. To measure this, it proved possible to acquire an old geometrician's theodolite, an instrument for measuring relative angles

[3] (Hamper, 2014)

[4] (Heavens-Above GmbH, 2015).

[5] *ISS Detector* (RunaR, 2015).

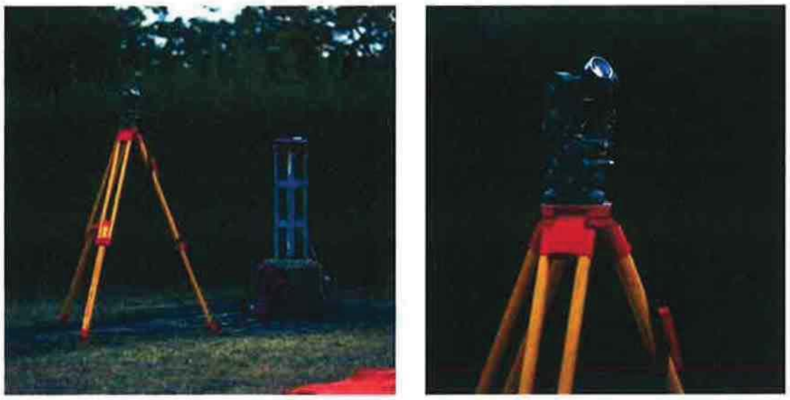

Figure A – *Apparatus*

at a distance (Figure A). The North Star was used to calibrate the instrument's azimuthal (i.e. horizontal) zero to the north. Vertical angles correspond to the angle between the zenith and the I.S.S. A spirit level was used to ensure azimuthal angles were parallel to the horizontal plane. The angles were given in grades, as those are the units used by that theodolite, an analogue device with, therefore, a fixed set of units.

The theodolite's aiming mechanism comprised a telescopic sight (see the second part of Figure A) as well as a rudimentary iron sight.[6] As it was not always possible to see the I.S.S. in the telescopic sight, the iron sight was occasionally used to estimate the accuracy of the measurement. The uncertainties were assigned through this method, giving $1{,}9^g$ (the angular width of the telescopic sight) if glimpsed through the lens, 5^g for a near miss and 10^g for an approximate bearing.

Safety and ethical considerations

The experiment in itself was generally innocuous. It was important to avoid tripping over the tripod and not fall asleep. We therefore granted ourselves appropriate rest breaks. Some caution was however necessary as the experiment had to be carried out at night. Because of this, I solicited two friends of mine at different times to keep me company and help me with the experiment, without whom this report would not have been possible. Furthermore, in order to avoid being stopped by the police for trespassing, we had to ask the *Observatoire de Lausanne* for permission to use their grounds, which was kindly granted.

Raw data and processing

Following is a table of all the data collected during the experiment. The first four columns give the time at which each sighting was taken. As we did not take down the times at which the chronometer was started or stopped, the *Approx. time* given is the time at which the satellite was predicted to be visible by the online data base. The angular position θ_n is where the I.S.S.

[6] "Iron sights are a system of shaped alignment markers (usually metal) used as a sighting device to assist in the aiming of a device such as a firearm, crossbow, or telescope, and exclude the use of optics as in telescopic sights or reflector (reflex) sights." (Wikimedia Foundation, Inc., 2015)

was in the sky when we took the reading n. The time T is the time between sightings as measured by our chronometer. Its uncertainty is arbitrarily set at an estimated twenty seconds.

The I.S.S. is not visible every night, and is not always visible in the same location in the sky. Furthermore, despite the high frequency of sightings (up to 5 per night on some nights) for which we selected the months of July-August 2015, a number of sightings had to be discarded due to contrary weather conditions, low visibility, or an insufficiently long visible time-frame. Because of this, we could not use all of the sightings we took. Those below are the ones that are of sufficiently good quality to be used in calculations.

Raw data:

Table 1 – ***Data gathered during the experiment***

Sighting				*Angular position [grads]*						*Time [seconds]*	
A		*B*		ϑ_A			ϑ_B				
Date	*Approx. time*	*Date*	*Approx. time*	*Az*	*V*	$\Delta\vartheta$	*Az*	*V*	$\Delta\vartheta$	*T*	ΔT
28/07/2015	02:23:00 AM	28/07/2015	03:58:00 AM	5,5	62,7	±10	1,6	58,1	±10	5682	±20
29/07/2015	01:34:00 AM	29/07/2015	03:09:00 AM	62,7	72,3	±10	10,8	68,9	±10	5762	±20
30/07/2015	10:09:00 PM	30/07/2015	11:43:00 PM	163,5	85,7	±10	241,8	83,6	±5	5595	±20
30/07/2015	11:43:00 PM	31/07/2015	01:25:00 AM	241,8	83,6	±5	55,5	72,4	±10	6078	±20
31/07/2015	01:25:00 AM	31/07/2015	02:58:00 AM	55,5	72,4	±10	5,9	58,3	±1.9	5664	±20
31/07/2015	02:58:00 AM	31/07/2015	04:35:00 AM	5,9	58,3	±1.9	287,7	46,6	±5	5770	±20
05/08/2015	09:37:00 PM	05/08/2015	11:15:00 PM	0,0	16,4	±10	0,0	65,5	±1.9	5808	±20
06/08/2015	10:22:00 PM	07/08/2015	12:00:00 AM	0,0	60,9	±1.9	394,5	56,5	±5	5773	±20
07/08/2015	09:29:00 PM	07/08/2015	11:06:00 PM	0,0	52,4	±5	0,0	64,8	±0.5	5793	±20

Simple model analysis:

In the simple scenario of a perfectly circular orbit, the values for the radius of orbit r and the linear velocity $\vec{v}$ of the I.S.S. around the earth are given by the system of equations presented in the section on THEOREMS. These are:

$$\vec{\omega} = \frac{2\pi}{T} \quad \text{and} \quad r = \sqrt[3]{G\frac{m_\oplus}{\vec{\omega}^2}} \quad \text{and} \quad \vec{v} = \sqrt[3]{G\vec{\omega}m_\oplus}$$

Having formulæ for radius r and velocity $\vec{v}$, we may consider the uncertainty in these values through error propagation. There is an error ΔT in the period of orbit T, estimated previously at approximately 20 seconds. The mass and radius of the Earth, respectively equal to $5{,}9726 \cdot 10^{24}$ kilogrammes and 6'371 kilometres,[7] are taken as exact, their respective percentage uncertainties being minute with respect to those resulting from the experiment. The gravitational constant is similarly taken to be exact with a value of $6{,}67384 \cdot 10^{-11}$ $m^3\ kg^{-1}\ s^2$, [(8)] rounded to two decimal places for the calculations. Changes in the altitude of the I.S.S., in the mass of the earth or in the position of the theodolite can be ignored as they are proportionally insignificant.

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[7] (N.A.S.A., 2013)

[8] *(Ibid.)*

The fractional error in the angular velocity is the same as that in the period of orbit, as 2π is a constant; this gives $\frac{\Delta\vec{\omega}}{\vec{\omega}} = \frac{\Delta T}{T}$. In order to calculate the uncertainty in the radius of orbit r and linear velocity $\vec{v}$, both r and $\vec{v}$ must be expressed as a sum, product or power of exact and inexact values;[9] in other words, in a form similar to $k_1 a^m b^n \pm k_2 c^p$. In this case, we get the form $k \cdot a^n$ for both r and $\vec{v}$:

$$r = \sqrt[3]{Gm_{\oplus}} \cdot \vec{\omega}^{-\frac{2}{3}} \text{ and } \vec{v} = \sqrt[3]{Gm_{\oplus}} \cdot \vec{\omega}^{\frac{1}{3}}$$

Applying the formulæ for error propagation, we get that the fractional uncertainties $\frac{\Delta\vec{v}}{\vec{v}}$ and $\frac{\Delta r}{r}$ in the linear velocity and radius of orbit respectively, are given by multiples of the fractional uncertainty $\frac{\Delta\vec{\omega}}{\vec{\omega}}$ in the angular velocity, as shown below. These are the equations which will be used to give the fractional uncertainties within the calculations.

$$\frac{\Delta r}{r} = \frac{2}{3}\frac{\Delta\vec{\omega}}{\vec{\omega}} = \frac{2}{3}\frac{\Delta T}{T} \text{ and } \frac{\Delta\vec{v}}{\vec{v}} = \frac{1}{3}\frac{\Delta\vec{\omega}}{\vec{\omega}} = \frac{1}{3}\frac{\Delta T}{T}$$

Noting, of course, that the values for the altitude and its respective uncertainty may be acquired from the following formulæ:

$$h = r - r_{\oplus} \text{ and } \Delta h = r\frac{\Delta r}{r}$$

This allows the creation of the following table (Table 2).

Processed data for first model:

Table 2 – *Results of the preliminary calculations*

Time [seconds]		*Angular velocity [radians per second]*		*Results of orbital calculations*					
				Radius [m]		*Height [km]*		*Velocity [m/s]*	
T	$\pm\Delta T$	ω	$\pm\Delta\omega$	r	$\pm\Delta r$	h	$\pm\Delta h$	v	$\pm\Delta v$
5682	20	0,001106	0,000004	6'882'200	16'150	511	16	7'610	9
5762	20	0,001090	0,000004	6'946'649	16'075	576	16	7'575	9
5595	20	0,001123	0,000004	6'811'769	16'233	441	16	7'650	9
6078	20	0,001034	0,000003	7'198'361	15'791	827	16	7'441	8
5664	20	0,001109	0,000004	6'867'658	16'167	497	16	7'618	9
5770	20	0,001089	0,000004	6'953'077	16'067	582	16	7'571	9
5808	20	0,001082	0,000004	6'983'571	16'032	613	16	7'555	9
5773	20	0,001088	0,000004	6'955'487	16'064	584	16	7'570	9
5793	20	0,001085	0,000004	6'971'542	16'046	601	16	7'561	9

Limitations of the simple model:

Unfortunately, the I.S.S. does not orbit in a perfect circle around the earth, as mentioned previously. This signifies that the orbital velocity of the I.S.S. was slightly bigger than

[9] (International Baccalaureate Organisation, 2014)

that given by the calculations above, as it travelled a path longer than that expected, in the same amount of time. This difference can, however, be accounted for through more rigorous analysis using a more complex model.

More complex modelling

In reality, the I.S.S. moves latitudinally over the course of its orbit. This scenario is illustrated in Figure B, which depicts the situation of the second reading, where the I.S.S. was initially visible to the north, then south on the second pass.

Because the change in latitude of the I.S.S. is marginal in comparison to the size of the earth, the earth can be approximated to a cylinder whose radius is that of the I.S.S.'s orbit and whose central axis lies on the perpendicular to, and passes through the centre of, the I.S.S.'s orbital plane (see Figure B). This is in order to simplify the mathematical aspect of the model. It is important to note that this approximation is only valid for small values of latitudinal displacement, when the curvature of the earth is minimally perceptible. Furthermore, it is assumed that the altitude of the I.S.S. remains that which was calculated in the previous section.

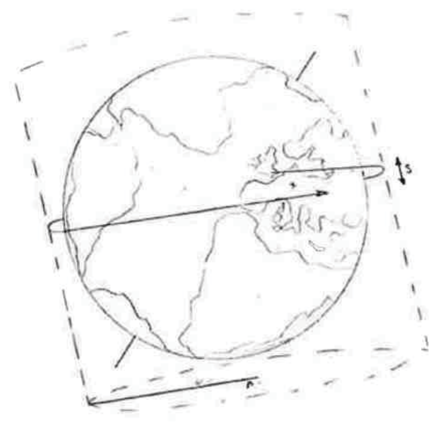

Figure B – ***Orbital Discrepancy***

Taking the net of the cylinder, ignoring the caps, a rectangle is produced onto which can be mapped the I.S.S.'s trajectory, letting the latitudinal displacement of the I.S.S. be represented by the symbol s (as shown in Figure B and Figure C).

This being a cylinder of the same radius as that of the trajectory of the I.S.S., the circumference of the circular section – that is to say, the width of the net in Figure C – is given by the simple expression $2\pi r$. From this, the actual path length of the I.S.S.'s orbit may be derived via the Pythagorean Theorem. It is:

$$\sqrt{4\pi^2 r^2 + s^2}$$

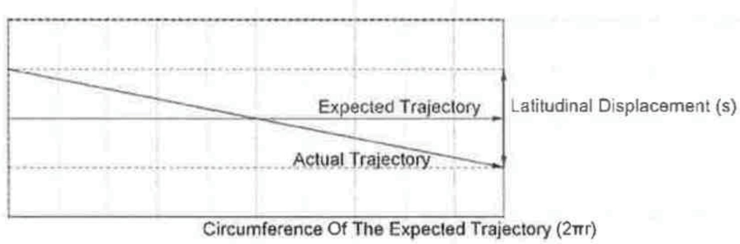

Figure C – ***Trajectory Compensation***

Taking the definition of velocity (i.e. displacement over time), this length must be equal to the product of the true velocity of the I.S.S., $\vec{v}_r$, by the time taken to complete a full orbit, which is the period T; in symbols, this is given by the following expression. The subscript r, for *rectified*, is used to represent a value belonging to the more complex model.

$$\vec{v}_r T = \sqrt{4\pi^2 r^2 + s^2}$$

In order for this to yield a consequential result, the latitudinal displacement must be calculated. For this, a cross-section of the earth through the latitude of the observer is taken. This allows us to apply trigonometry to the question of calculating the latitudinal displacement from the respective angles of the culmination of the I.S.S.'s different passes. Figure D shows this cross-section for two situations. In the first (I), the I.S.S. first appears to the north, then to the south of the observer. In the second (II), both sightings are to the north of the observer.

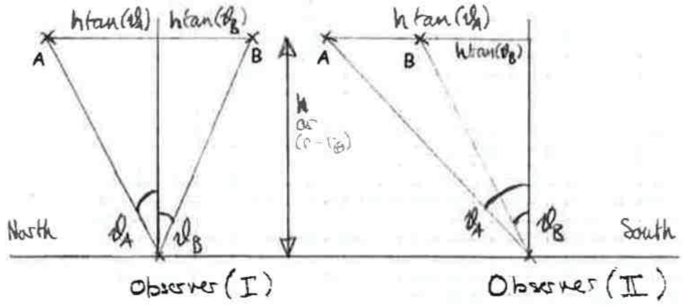

Figure D – *A cross-section of the earth through the latitude of the observer*

The latitudinal displacement is the distance between points *A* and *B*; thus, it is given by the magnitude of the difference between the lengths $(r - r_{\oplus})\tan(\theta_A)$ and $(r - r_{\oplus})\tan(\theta_B)$. Mathematically, this is:

$$s = |(r - r_{\oplus})[\tan(\theta_A) - \tan(\theta_B)]|$$

The uncertainty in $\vec{v}_r$ has been calculated to be the following. For an explanation of the derivation of this formula, the reader is invited to consult *Appendix a) Uncertainties in the complex model* (p. XVII).

$$\frac{\Delta\vec{v}_r}{\vec{v}_r} = \frac{s^2\frac{\Delta s}{s} + \frac{8}{3}\pi^2 r^2 \frac{\Delta T}{T}}{4\pi^2 r^2 + s^2} + \frac{\Delta T}{T}$$

Where:

$$\frac{\Delta s}{s} = \frac{2r\Delta T}{3T(r - r_{\oplus})} + \left|\frac{|\Delta\theta_A \sec^2(\theta_A)| + |\Delta\theta_B \sec^2(\theta_B)|}{\tan(\theta_A) - \tan(\theta_B)}\right|$$

From this, we can obtain the following table.

Processed data for second model:

Table 3 – *Results of the more complex modelling*

Time [seconds]		*Initial calculations: results [m]*			*Vertical, angular measurement [radians]*				*Secondary calculations & results*				
		Radius		*Height*	*A*		*B*		$\Delta\tau$	*Latitudinal disp.*		*Rect. velocity [m/s]*	
T	$\pm\Delta T$	r	$\pm\Delta r$	h	ϑ_A	$\pm\Delta\vartheta_A$	ϑ_B	$\pm\Delta\vartheta_B$		s [m]	$\Delta s/s$	v_r	Δv_r
5682	20	6,882,200	16,150	511,200	0.985	0.157	0.913	0.157	0.93	109,116	4.41	7,610	45
5762	20	6,946,649	16,075	575,649	1.136	0.157	1.082	0.157	1.60	155,337	5.95	7,575	44
5595	20	6,811,769	16,233	440,769	-1.346	0.157	-1.313	0.079	4.38	256,167	7.57	7,650	48
6078	20	7,198,361	15,791	827,361	-1.313	0.079	1.137	0.157	2.10	4,927,606	0.37	7,485	73
5664	20	6,867,658	16,167	496,658	1.137	0.157	0.916	0.030	0.97	426,343	1.16	7,619	46
5770	20	6,953,077	16,067	582,077	0.916	0.030	-0.732	0.079	0.22	1,280,757	0.13	7,575	45
5808	20	6,983,571	16,032	612,571	0.258	0.157	1.029	0.030	0.28	856,087	0.23	7,556	44
5773	20	6,955,487	16,064	584,487	0.957	0.030	0.887	0.079	0.29	110,940	1.54	7,570	44
5793	20	6,971,542	16,046	600,542	0.823	0.079	1.018	0.008	0.20	325,501	0.39	7,562	44

Results: discussion

The values obtained may now be compared to the literature values of the quantities; for the velocity, we will also consider the simple *vs* the complex model (or Table 3 p. X against Table 2 p. VIII).

Altitude:

The more complex modelling does not re-evaluate the altitude of the I.S.S., so the results for the altitude come only from the preliminary calculations. They are less accurate, although they are of the correct order of magnitude. For the period of July to August, the real altitude of the I.S.S. fluctuated between 401'000 and 403'000 metres.[10] The calculated results are all much greater than this, the smallest being thirty-eight kilometres above the four-hundred-and-two kilometre mark (as can be seen in Figure E). I conclude that there must therefore be a systematic error in the altitude.

As $(h + r_\oplus) \propto T^{\frac{2}{3}}$, and $r_\oplus$ is constant, over-estimating the period of orbit would have led to an over-estimation of the altitude. As this is what has been seen in our figures, evidence therefore strongly suggests that the period of orbit of the I.S.S. was consistently over-estimated, causing the systematic offset in the altitude, although I do not know why.[11]

Please note that error bars are included in Figure E. However, they are too small to visibly appear on the graph.

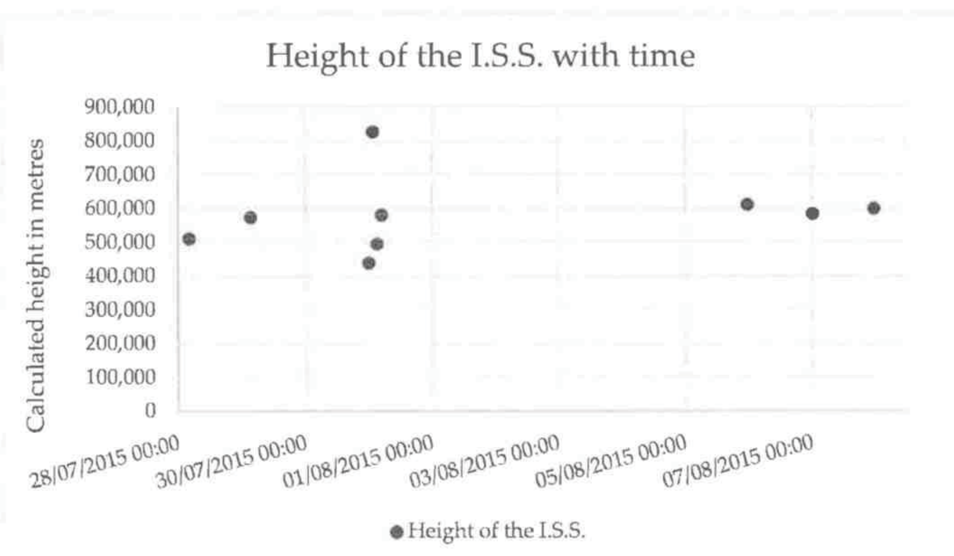

Figure E – ***Calculated altitude of the I.S.S. with time (graph)***

[10] (Heavens-Above GmbH, 2015) (See second appendix.)

[11] Possibilities for improvements to avoid such incoherencies are explored in the *Evaluation of experimental method & observations* below.

Interestingly, the two data below the 500'000-metre mark appear to coincide with two occurrences where the azimuthal bearing strayed further away from the meridian[12] (refer to Table 1, on page VII). Whether or not there is a correlation, and why, could provide a basis for a further investigation in more depth.

Before concluding, we will experiment with modifying the uncertainty in the period of orbit. If this is granted more importance and raised from twenty seconds to a minute and a half, it has the following effect, visible in Table 4 and illustrated in Figure F.

Table 4 – ***Reappraisal of uncertainty on time***

Time [seconds]		*Initial calculations: results [m]*			*Vertical, angular measurement [radians]*				*Secondary calculations & results*				
		Radius		*Height*	*A*		*B*		$\Delta\tau$	*Longitud. disp.*		*Rect. velocity [m/s]*	
T	$\pm\Delta T$	r	$\pm\Delta r$	h	ϑ_A	$\pm\Delta\vartheta_A$	ϑ_B	$\pm\Delta\vartheta_B$		s [m]	$\Delta s/s$	v_r	Δv_r
5682	210	6'882'200	169'572	511'200	0,985	0,157	0,913	0,157	0,93	109'116	4,71	7'610	469
5762	210	6'946'649	168'784	575'649	1,136	0,157	1,082	0,157	1,60	155'337	6,21	7'575	461
5595	210	6'811'769	170'446	440'769	-1,346	0,157	-1,313	0,079	4,38	256'167	7,92	7'650	481
6078	210	7'198'361	165'806	827'361	-1,313	0,079	1,137	0,157	2,10	4'927'606	0,55	7'485	478
5664	210	6'867'658	169'751	496'658	1,137	0,157	0,916	0,030	0,97	426'343	1,47	7'619	472
5770	210	6'953'077	168'706	582'077	0,916	0,030	-0,732	0,079	0,22	1'280'757	0,39	7'575	462
5808	210	6'983'571	168'337	612'571	0,258	0,157	1,029	0,030	0,28	856'087	0,48	7'556	457
5773	210	6'955'487	168'676	584'487	0,957	0,030	0,887	0,079	0,29	110'940	1,80	7'570	459
5793	210	6'971'542	168'482	600'542	0,823	0,079	1,018	0,008	0,20	325'501	0,65	7'562	457

Modifying the uncertainty in the period of orbit causes all the uncertainties for the altitude (except for the 827 km altitude, which remains anomalous) effectively to encompass the accepted literature values. Thus, the disagreement with literature values was most probably linked to an under-estimate of the error in the time; this is not unlikely, as the original twenty seconds were arbitrarily set.

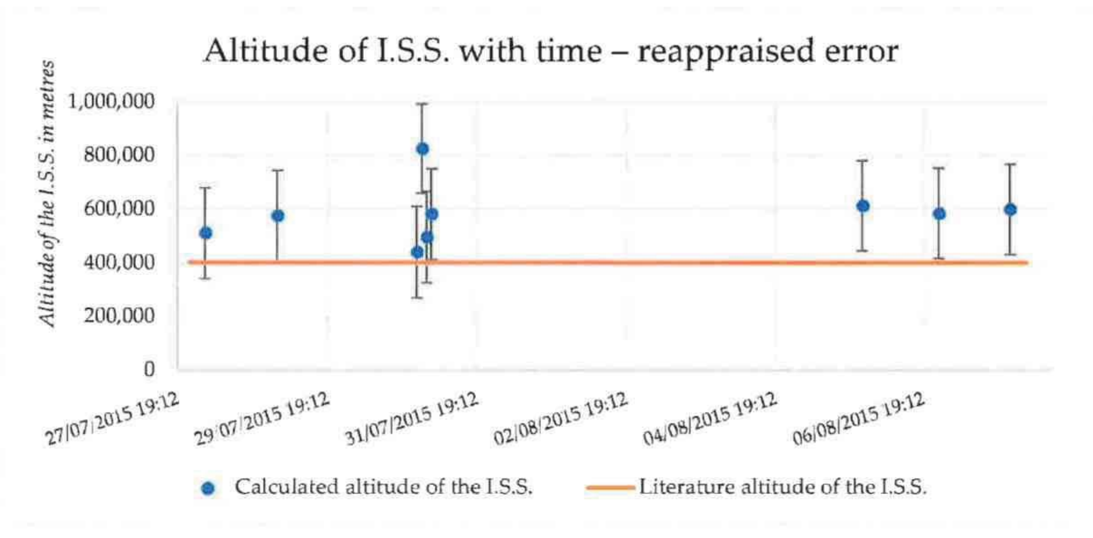

Figure F – ***Graph of the calculated altitude with revised errors***

[12] A meridian is the north-south line passing through an observer. It can also be referred to as the latitude of the observer.

Linear velocity:

The calculated linear velocity of the I.S.S. changes little from Table 2 (p. VIII) to Table 3 (p. X). This is probably due to the fact that, as the latitudinal displacement s is very small with respect to the earth's circumference, taking it into account changes little in terms of the actual path travelled by the I.S.S. Indeed, the difference in the values from Table 2 to Table 3 is generally of 1 m s^{-1} or less. This is the case for all values but one, the fourth in Table 3. This reading, however, also leads to the result that the I.S.S. is 827'000 metres above the ground, two hundred kilometres more than the second-largest. Thus, it can plausibly be discounted as an outlier, most probably due to the chronometer failing to be activated at the correct time.

The accuracy of the calculated velocity is surprisingly good, far more so than that of the altitude. The literature value for the velocity of the I.S.S. is an average of approximately 7'660 metres per second, with no uncertainty provided (Wikimedia Foundation, Inc., 2015). Most of the calculated velocities (saving the outlier) fall within three per cent of this, an excellent experimental result.[13] However, the uncertainties are not always quite sufficient to include the literature value. They tend towards being 50 rather than the necessary 100–200 m s^{-1}. However, this issue is resolved by the re-evaluation of the uncertainty in the period.

It is interesting to note that the largest value for velocity within each datum's original uncertainty range (i. e. $\vec{v}_r + \Delta\vec{v}_r$) tends to fall *short* of 7'660 m s^{-1}. This is the case in six of the eight non-outlier scenarios (in the other two, the sum of result and uncertainty includes 7'660, but the result itself is still smaller than that). Perhaps this is due to the fact that, having ignored the latitudinal curvature of the earth, we have over-estimated the distance travelled by the I.S.S. This would increase the velocity, however,[14] so it cannot be the cause of this tendency. The alternative is that the time has somehow been consistently over-estimated, due to a systematic error in either methodology or calculations.

All in all, the greatest issue, that of literature values falling outside the uncertainty range, seems to be due to having under-estimated the error in the time, as mentioned previously; this does not however solve the systematic over- and under-estimates of altitude and velocity, though it does show that the calculations and methods used were sound.

EVALUATION AND CONCLUSION

Evaluation of experimental method & observations:

To predict the altitude and velocity of the I.S.S., we have used a relatively simple method which involved taking the position of the I.S.S. using a theodolite, and starting and stopping a clock at the same time as the readings were taken. There were some weaknesses with this method. The most important of these is the fact that the satellite did not necessarily pass

[13] Calculation is largest deviation from value divided by the value: $\frac{7660-7482}{7660} = 0.026 \ldots \cong 3\%$.

[14] Since $v = \frac{d}{t}$.

through diametrically opposed positions on consecutive sightings. The more complex cylindrical approximation used in the section on further modelling functions on the principle that sightings are taken when the I.S.S. is either at the same compass bearing or at diametrically opposite positions. However, we only realised the importance of this only in August. Ideally, the entirety of the experiment would have been re-run with this modification but due to constraints on time, it would not then have been possible to amass sufficient data. Thus, only the data from August 5th onwards attempt to use a consistent line (in this case the meridian) at which to measure the I.S.S.[15] This is suboptimal but not catastrophic, as in any case the errors in time and angle due to this are minor; my estimates place them at around five minutes and 20-to-25 grades.

In the future, achieving a consistent reference line can be achieved by locking the theodolite's horizontal movement on one position. The theodolite would consider one side (e.g. north) a positive vertical angle V, and the other, a negative angle. Applying this modification would allow more consistency and improve the precision of the angular reading by making it easier to catch the moving I.S.S. with the lens (as, instead of trying to track a moving object in two orthogonal angles, one only has to wait until the I.S.S. crosses the reference line and can pre-empt it by placing the theodolite at roughly the right vertical angle).

Visible I.S.S. passes did not always cross the meridian, which was the reference line we used for the datum-gathering; ideally, those passes would have been ignored, as they do not conform perfectly to the second model's assumption that the earth can be modelled by a cylinder. Because of time constraints, they were not. However, the afore-mentioned modification of 'azimuthal locking' could also maximise the number of useable sightings, if the reference line picked coincides with a line most visible I.S.S. passes cross.[16]

The way we measured the period was by having one chronometer activated at a fly-by, stopped at the next, read off, and then activated again for the next orbit. This could be improved by having, instead, two chronometers alternating at every sighting (so that one can start as another stops), as well as by being particularly attentive to make sure the chronological and angular readings are synchronised. This change might help reduce uncertainty and inaccuracy in the resultant period. [17]

Another potential source of error is that the telescopic sight was very awkward to look through at very steep angles. This was circumvented by rotating it through a full 200 grades using an improvised parallel guide (a pen or pencil pinned to the theodolite, parallel to the original bearing), after which the angle could be read off the theodolite. In this report, the

[15] The point at which the I.S.S. is on the observer's meridian is also known as its astrological culmination point (Oxford University Press, 1982 p. 231).

[16] This works because it does not matter to the cylindrical approximation whether the cylinder is parallel or not to the earth's north-south axis, so long as the 'start' and 'stop' points of the chronometer have the I.S.S. lying on some common line approximately perpendicular to the I.S.S.'s orbital path.

[17] One could potentially also have multiple chronometers going simultaneously to simulate repeats of the experiment, but I believe this is redundant, as error due to e.g. atmospheric friction would be far greater than the accuracy afforded by multiple chronometers, making them somewhat redundant.

uncertainty was increased when this occurred, to account for potential error. In the future, error could be prevented through the use of an elbow, a module attached to the eye-piece to redirect light into the eye of the observer, thus allowing the reading of steeper angles.

Some error may have been present in the calibration to north, as it was occasionally necessary to use land-marks due to the North Star not being visible. Imperfect horizontality in the instrument's azimuthal plane was rendered improbable through the use of a level. Finally, certain sightings were unusable for the final results due to contrary factors, such as poor visibility due to weather.

Evaluation of models:

Increasing the complexity of the model has produced more sensible uncertainties for the results of the velocity. However, an even more detailed analysis may consider a few more elements. It might include the latitudinal curvature of the earth, in contrast to the cylindrical approximation made use of here. At the same time, it might allow for the variation in altitude of the I.S.S. from one sighting to another and base its calculation on more factors than the space-craft's velocity. It might furthermore even take into account some of the following: changes in the position of the I.S.S. due to the refraction of light through the atmosphere; the gravitational interference from other satellites; the influence of air friction due to the shape of the I.S.S.; air currents and weather conditions; changes in the mass of the earth; the imperfectly spherical nature of the planet; and many more issues. These are, however, probably minute. I estimate none of them to amount to a difference in the results of more than 1 in 10^3 or 10^4 in magnitude for the velocity. They may be somewhat more substantial for the altitude, between 1 in 10^3 and 1 in 10^1. It would also be quite interesting to re-visit the experiment in order to understand the cause of the systematic offsets.

Conclusion:

Generally, the experiment was very successful. We have been able predict the altitude and velocity of the I.S.S. with a good degree of accuracy, in spite of the relative simplicity of the models and observations. The velocity has proven to be more accurate than the altitude, the former being accurate to almost two significant figures and the latter being correct to within half an order of magnitude. The largest source of error has proved to be due to an under-estimate of the uncertainty in the period of orbit, and the second is strongly suspected of being a systematic over-estimate of the period itself. The more complex modelling also effectively demonstrated the insignificance of errors in approximations (i.e. of the earth to a cylinder) when used for small values relative to the magnitude of the earth. All in all, the experiment has shown, in practice, the validity of simple theorems (such as the Laws of Gravitation and of Circular Motion) even when applied to complex situations; the answer to the research question, '*How can simple measurements of the orbital period of a satellite be used to calculate its altitude and velocity?*', is thus 'with good relative accuracy'.

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Appendices

a) Uncertainties in the complex model:

These are the calculations for the uncertainty in the rectified velocity $\vec{v}_r$, which is equal to:

$$\vec{v}_r = \frac{\sqrt{4\pi^2 r^2 + s^2}}{T}$$

Where:

$$s = |(r - r_{\oplus})[\tan(\theta_A) - \tan(\theta_B)]|$$

First, we must address the compound $[\tan(\theta_A) - \tan(\theta_B)]$. The I.B. Physics H.L. syllabus prescribes, for the uncertainties of trigonometric functions, the use of the maximum deviation of $[\tan(\theta_A \pm \Delta\theta) - \tan(\theta_B \mp \Delta\theta)]$ from $[\tan(\theta_A) - \tan(\theta_B)]$. However, we will prefer to use an alternative in order to facilitate the calculations: for a function $\tan(\theta)$, the uncertainty in the result is given by $\Delta\theta \cdot \sec^2(\theta)$ on the condition that all angles be articulated in radians.[18] Thus:

$$\Delta[\tan(\theta_A) - \tan(\theta_B)] = |\Delta\theta_A \sec^2(\theta_A)| + |\Delta\theta_B \sec^2(\theta_B)|$$

We will now refer to this as $\Delta\tau$, for simplicity. As the latitudinal displacement s is given by the expression $|(r - r_{\oplus})[\tan(\theta_A) - \tan(\theta_B)]|$, its fractional uncertainty is given by taking the sum of the fractional uncertainties of its factors. It thus corresponds to the following equation, where Δr has previously been calculated to be $\frac{2r}{3}\frac{\Delta T}{T}$.

$$\frac{\Delta s}{s} = \frac{\Delta r}{r - r_{\oplus}} + \left|\frac{\Delta\tau}{\tan(\theta_A) - \tan(\theta_B)}\right|$$

The combination of all these equations enables the calculation of the uncertainty in $\sqrt{4\pi^2 r^2 + s^2}$. We first use the law of uncertainty propagation with powers to give the fractional uncertainty in s^2, which is $2 \cdot \frac{\Delta s}{s}$, and in $4\pi^2 r^2$, which is $2 \cdot \frac{\Delta r}{r}$ or $\frac{4}{3} \cdot \frac{\Delta T}{T}$. Then, we convert them

[18] (Arboleda, et al., 2007)

into absolute uncertainties, respectively giving $2s^2 \cdot \frac{\Delta s}{s}$ and $4\pi^2 r^2 \cdot \frac{4}{3}\frac{\Delta T}{T}$. We then add them together to give the absolute uncertainty in $4\pi^2 r^2 + s^2$ to be $\left(2s^2 \frac{\Delta s}{s} + \frac{16\pi^2 r^2}{3}\frac{\Delta T}{T}\right)$. To calculate the fractional uncertainty in $\sqrt{4\pi^2 r^2 + s^2}$, one halves that of $4\pi^2 r^2 + s^2$, as the former is the latter elevated to the power of one half. Thus, the fractional uncertainty in the total displacement $\vec{v}_r T$ is half the fractional uncertainty in $4\pi^2 r^2 + s^2$, given that $\vec{v}_r T = \sqrt{4\pi^2 r^2 + s^2}$. Thus:

$$\frac{\Delta(\vec{v}_r T)}{\vec{v}_r T} = \frac{1}{2} \cdot \frac{2\, s^2 \frac{\Delta s}{s} + \frac{16}{3}\, \pi^2\, r^2 \frac{\Delta T}{T}}{4\pi^2 r^2 + s^2}$$

Given that $\vec{v}_r$ is the quotient of $\sqrt{4\pi^2 r^2 + s^2}$ by T, the fractional uncertainty in $\vec{v}_r$ may be obtained by taking the sum of the fractional uncertainties of its numerator and denominator. Thus:

$$\frac{\Delta\vec{v}_r}{\vec{v}_r} = \frac{s^2 \frac{\Delta s}{s} + \frac{8}{3}\pi^2 r^2 \frac{\Delta T}{T}}{4\pi^2 r^2 + s^2} + \frac{\Delta T}{T}$$

Where:

$$\vec{v}_r = \frac{\sqrt{4\pi^2 r^2 + s^2}}{T}$$

$$s = |(r - r_\oplus)[\tan(\theta_A) - \tan(\theta_B)]|$$

$$\frac{\Delta s}{s} = \frac{\Delta r}{r - r_\oplus} + \left|\frac{\Delta\tau}{\tan(\theta_A) - \tan(\theta_B)}\right|$$

$$\Delta\tau = |\Delta\theta_A \sec^2(\theta_A)| + |\Delta\theta_B \sec^2(\theta_B)|$$

$$r = \sqrt[3]{Gm_\oplus} \cdot \left(\frac{2\pi}{T}\right)^{-\frac{2}{3}}$$

$$\text{and } \frac{\Delta r}{r} = \frac{2}{3}\frac{\Delta T}{T}$$

b) Literature altitude of the I.S.S. over time:

The following is a graph taken from the web site *Heavens-Above*[19] giving the altitude of the I.S.S. in kilometres between early September 2014 and late August 2015. In the period of time, which interests us (highlighted), the I.S.S. tended to decrease in altitude from ~402,5 km in late July to ~401,5 km mid-August. This downwards trend was not visible in the results of the experiment (Figure F, p. XII) because of the relatively high uncertainty on the altitude.

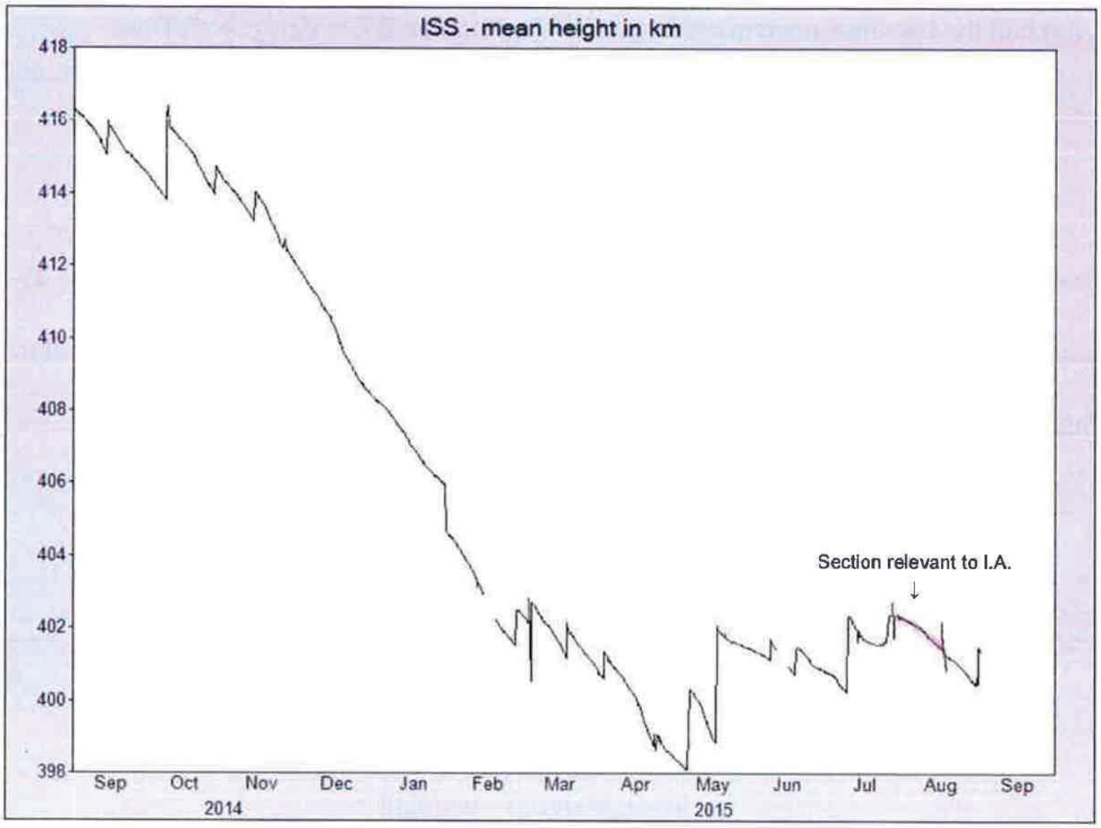

Figure G – ***Mean altitude of the I.S.S. in kilometres***

[19] (Heavens-Above GmbH, 2015)